THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial Questions for 13 Oct

1. (Average of a Sequence)

   Definition 1. *Let (a_n) be any sequence of real numbers. We define its partial sum*

$$S_n := \sum_{k=1}^{n} a_k,$$

   and then the average of it by

$$A_n := \frac{S_n}{n}.$$

   (a) Show that if $\lim_{n\to\infty} a_n = l \in \mathbb{R}$, then

$$\lim_{n\to\infty} A_n = l.$$

   (b) Show that the converse of (a) is not true by constructing a real sequence a_n whose average converges to a finite limit $l \in \mathbb{R}$ but a_n itself diverges.

   (c) Suppose in (b), we further assume that (a_n) is monotone, and that A_n converges to $l \in \mathbb{R}$. Then a_n converges to l.

2.

   Definition 2. *Let $f : [a, b] \to \mathbb{R}$ be a function of a real variable, where $a < b$. We say f is a contraction on $[a, b]$ if there exists $0 \le c < 1$ such that for any $x, y \in [a, b]$,*

$$|f(x) - f(y)| \le c|x - y|$$

   Consider a real sequence (x_n) defined via the recursive relation such that every x_n lies in $[a, b]$:

$$x_1 := x_0 \in \mathbb{R}, \qquad x_{n+1} := f(x_n), n \ge 1.$$

   (a) Let f be a contraction. Show that the sequence (x_n) converges to some $x \in [a, b]$ with $x = f(x)$.

   Hint:
   You may assume that f is continuous, hence by sequential criterion, if $x_n \to x$, then $f(x_n) \to f(x)$.

   (b) Let $x_0 = 2$ and let

$$f(x) := \frac{1}{2}(x + \frac{1}{x}).$$

   Show that (x_n) converges to 1.

   Hint:
   Step 1: Using mathematical induction, show that x_n defined in this way is monotonically decreasing, and that $1 \le x_n \le 2$ for all n.
   Step 2: Set $[a, b]$ to be $[1, 2]$ and show that f is a contraction on $[a, b]$.

(c) Show that the iterated square roots is convergent, and compute its limit:

$$l := \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

Hint:
Define $f(x) := \sqrt{x + 2}$ on some domain $[a, b]$.